CS103
Winter 2026

Lecture 06:
Functions

Part 1 of 2

Functions

Part 1

1. Announcements
2. Functions: Intuitions and Examples
3. Domains and Codomains
4. Official Rules for Functions
5. Ways to Define Functions
6. Involutions (with Proofs!)
7. The Assume-Prove Table
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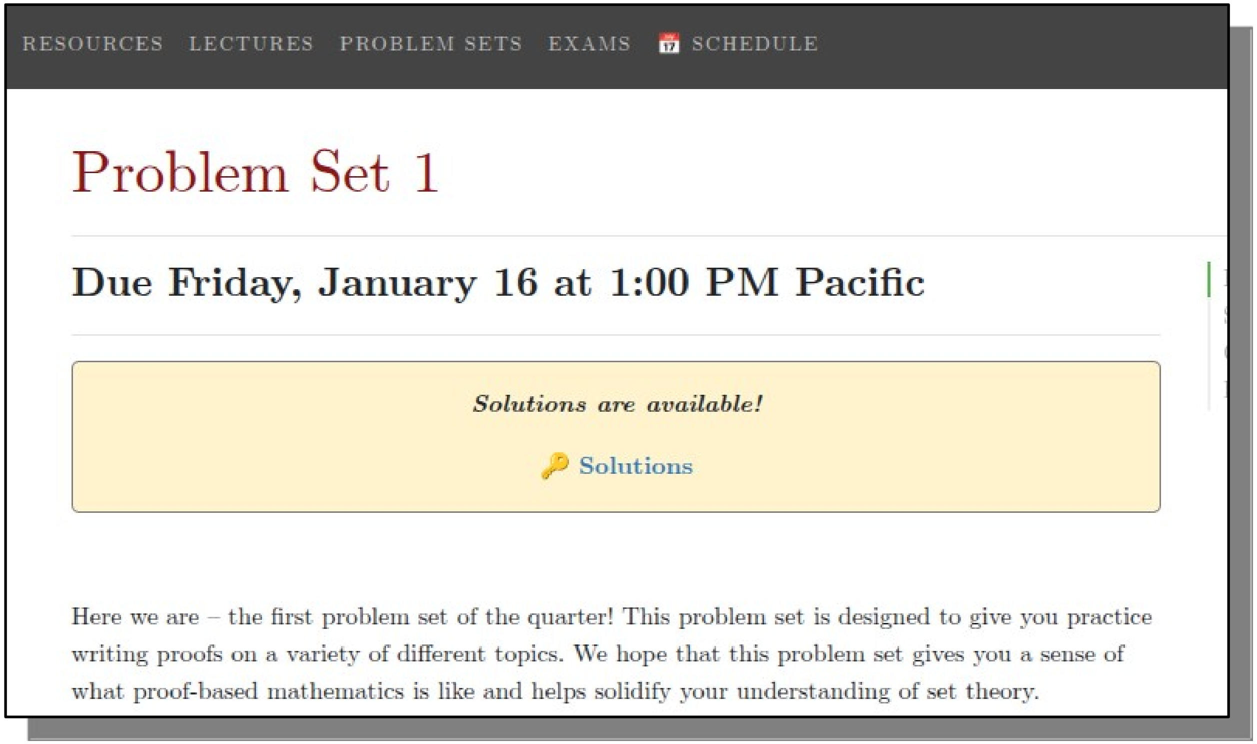
Problem Set One Graded

- TAs have finished grading Problem Set One!
- Grades and feedback are up on the Gradescope.
- Solutions posted on course website (via PS1 page). Check them out!

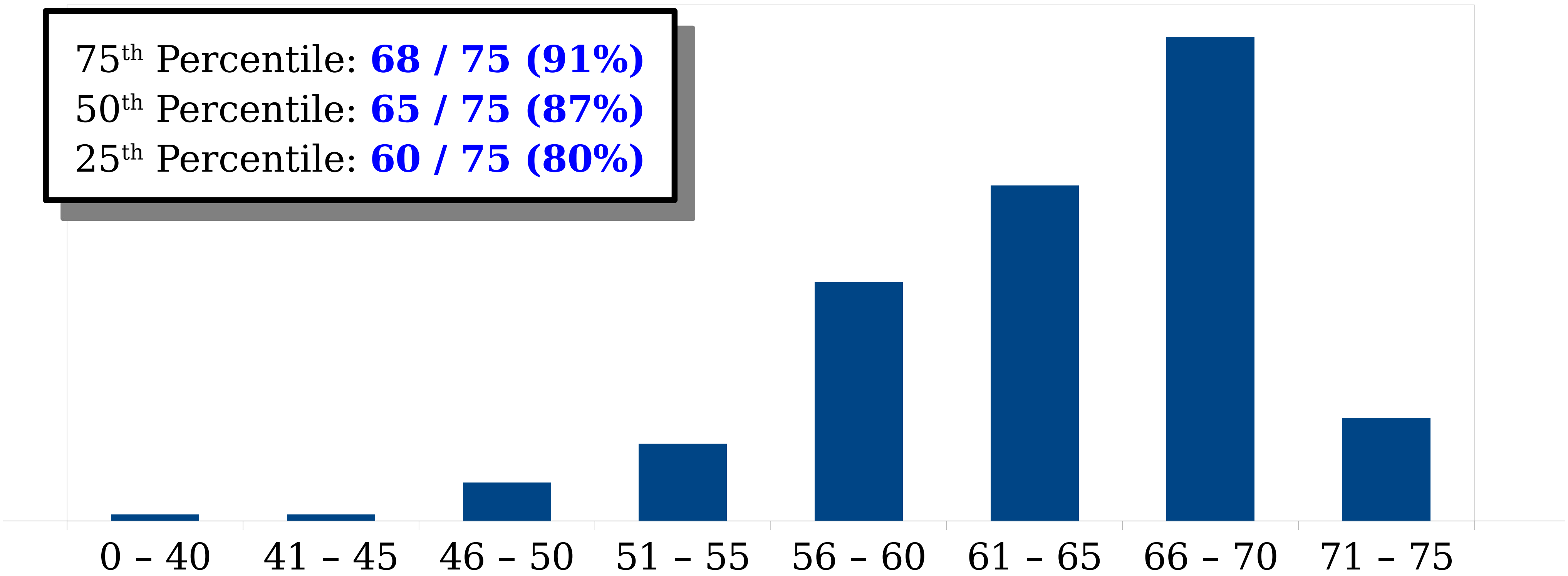


Problem Set One Graded

- TAs have finished grading Problem Set One!
- Grades and feedback are up on the Gradescope.
- Solutions posted on course website (via PS1 page). Check them out!
  - You'll get to see examples of polished written proofs.
  - We may have solved the problem differently than you, and this will give you more perspectives to use.
  - Each problem has a "Why We Asked This Question" section, which gives some context.

Problem Set One Graded

75th Percentile: **68 / 75 (91%)**
50th Percentile: **65 / 75 (87%)**
25th Percentile: **60 / 75 (80%)**



Pro tips when reading a grading distribution:

1. Standard deviations are ***unhelpful and discouraging***. Ignore them.
2. The average score is a ***unhelpful***. Ignore it.
3. Raw scores are ***unhelpful and discouraging***. Ignore them.

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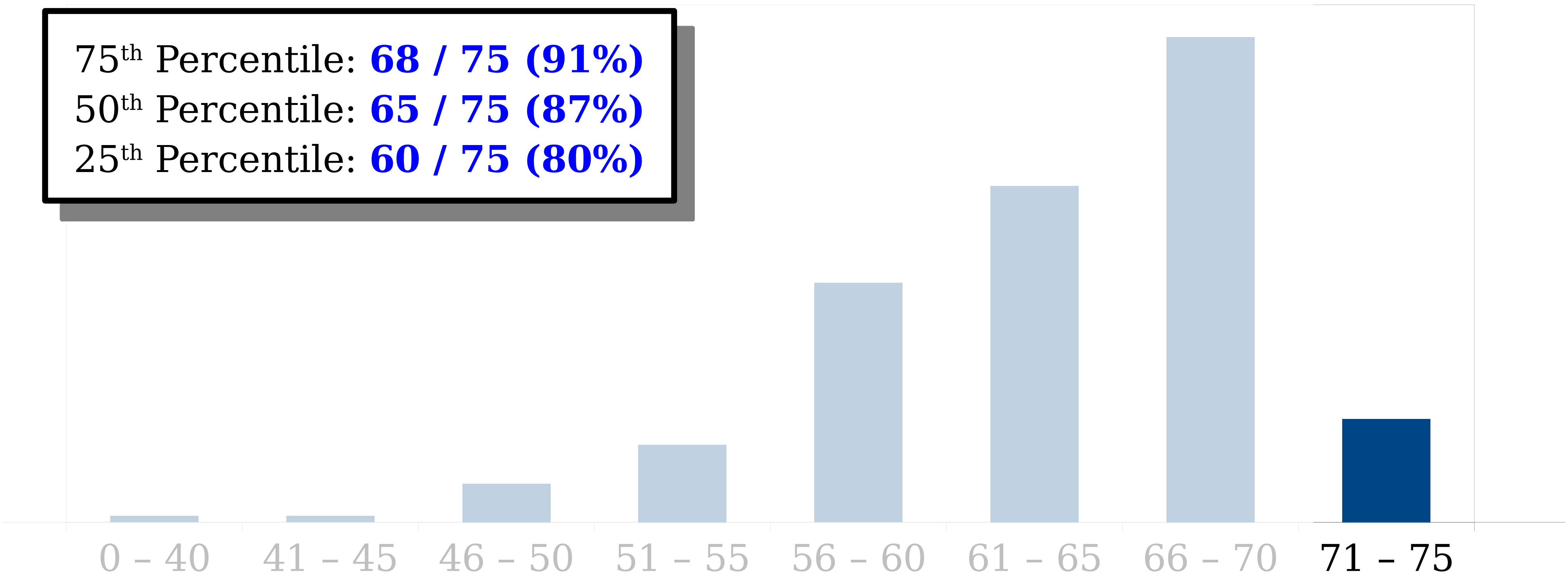

"Great job! Look over your feedback for some tips on how to tweak things for next time."

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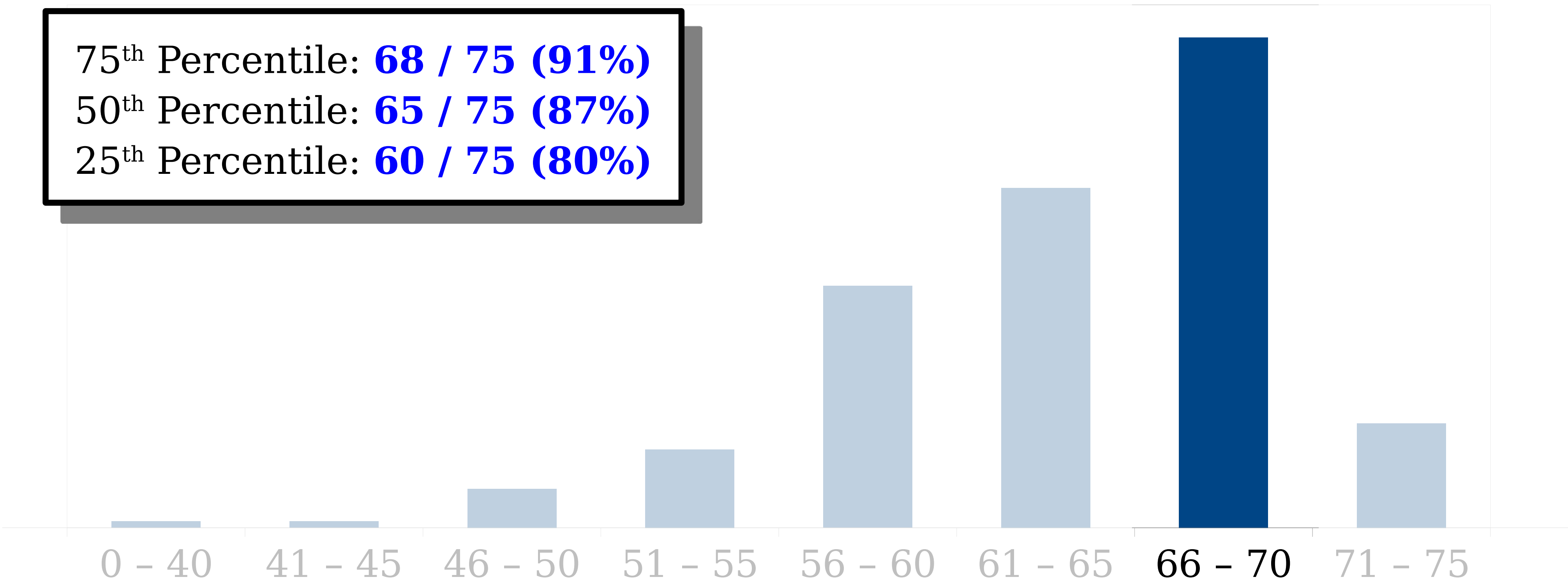

"You're almost there! Review the feedback on your submission and see what to focus on for next time."

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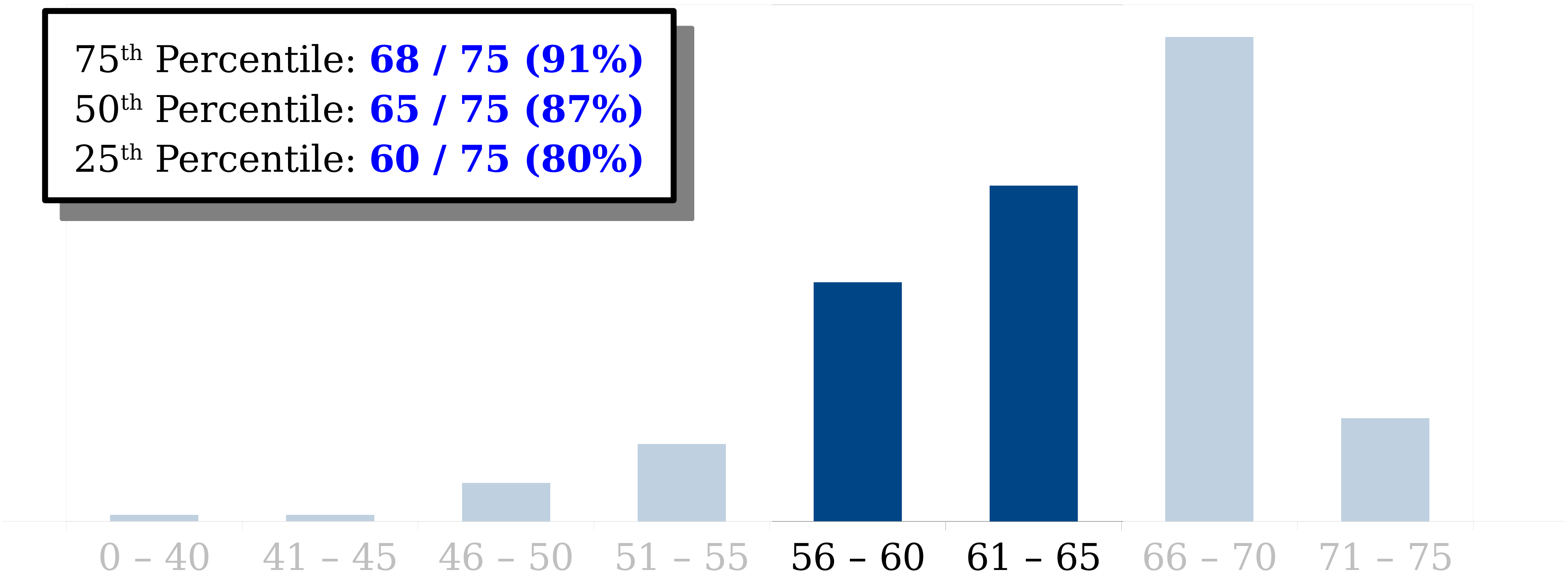

"You're on the right track, but there are some areas where you need to improve. Review your feedback and ask us questions when you have them."

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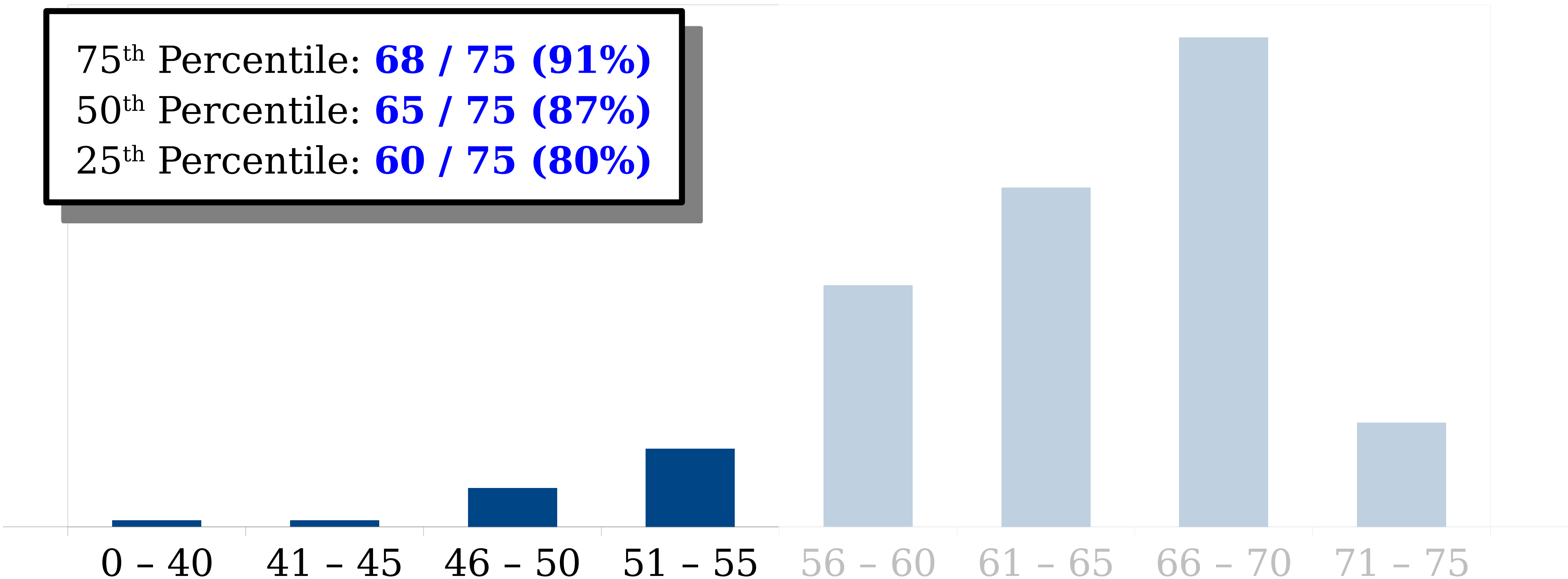

"Looks like something hasn't quite clicked yet. Get in touch with us and stop by office hours to get some extra feedback and advice. Don't get discouraged – you can do this!"

What Not to Think

- "Well, I guess I'm just not good at math."
  - For most of you, this is your first time doing any rigorous proof-based math.
  - Don't judge your future performance based on a single data point.
  - Life advice: adopt a growth mindset!
- "Hey, I did above the median. That's good enough."
  - There's always some area where you can improve. Take the time to see what that is.

Regrade Requests

- We’re human. We make mistakes. And we’re happy to correct them!
- Regrades general open on Gradescope 48 hours after grades are released and close one week after grades are released.
- Notes on regrades:
  - Please be civil. We make mistakes. We’re happy to correct them.
  - We have to grade what you submitted; we can’t take any clarifications into account during regrades.
  - Regrades are for where we made deductions we shouldn’t have, rather than for the magnitude of deductions.

Essential Action Items

- ***Review your feedback when it comes available.***
  - Don't just look at the raw score. Make sure you really, truly understand where you need to improve.
- ***Read the solutions in depth.***
  - Make sure you understand what we were asking, why we asked it, and what we wanted you to take away.
  - (Especially for Q8, Q10) Look at our solutions and see if there's any neat lessons you can draw from them.
- ***Come to us with questions.***
  - Anything you're not sure about? That's what we're here for! Come to office hours, ask questions on EdStem, etc.

Other Things to Have On Your Radar

- ***Left-handed desk form***
  - due Monday of next week
- ***Attendance opt-out form***
  - available sometime next Monday, due next Friday
- ***Regret Clause Forms***
  - due Tuesdays at 1:00 PM

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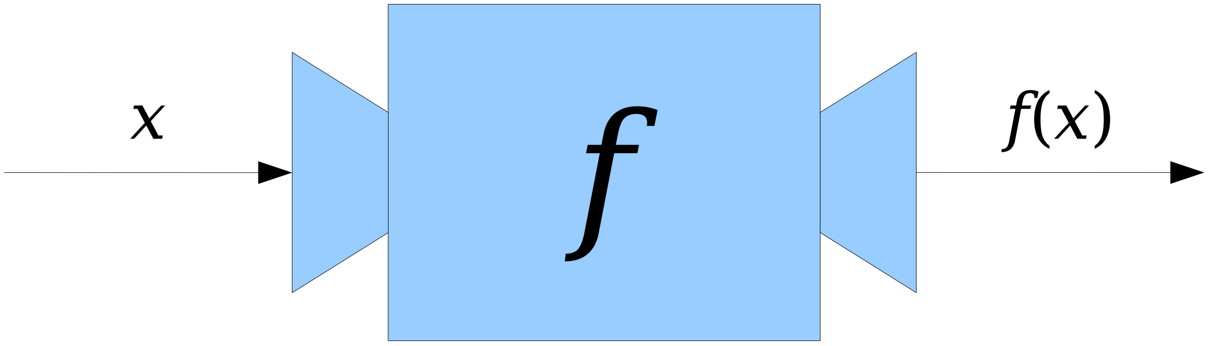
What is a function?

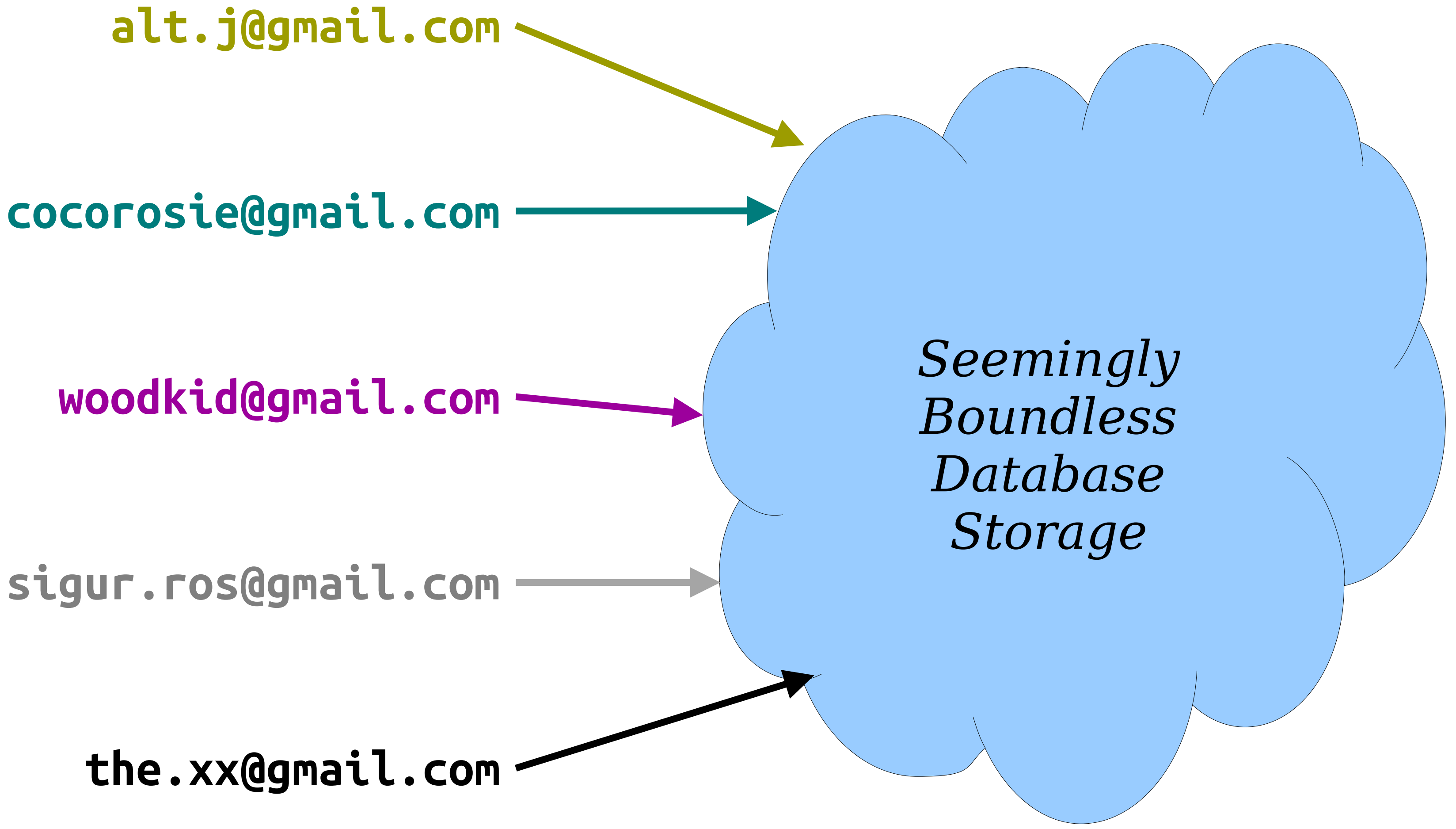
High-Level Intuition:

A function is an object f that takes in exactly one input x and produces exactly one output $f(x)$.



(This is not definition. It's just to help you build and intuition.)

Example: Database Sharding



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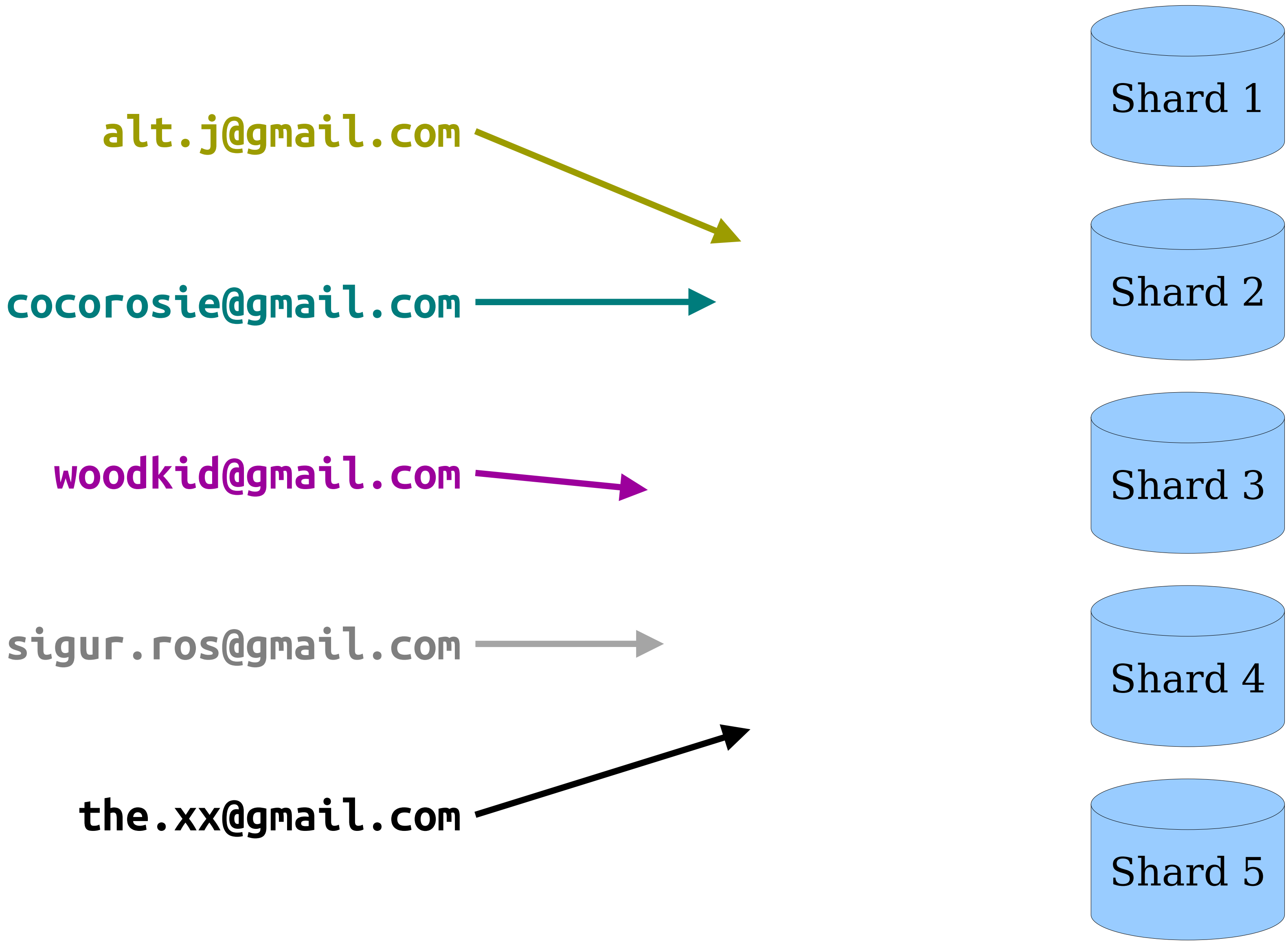

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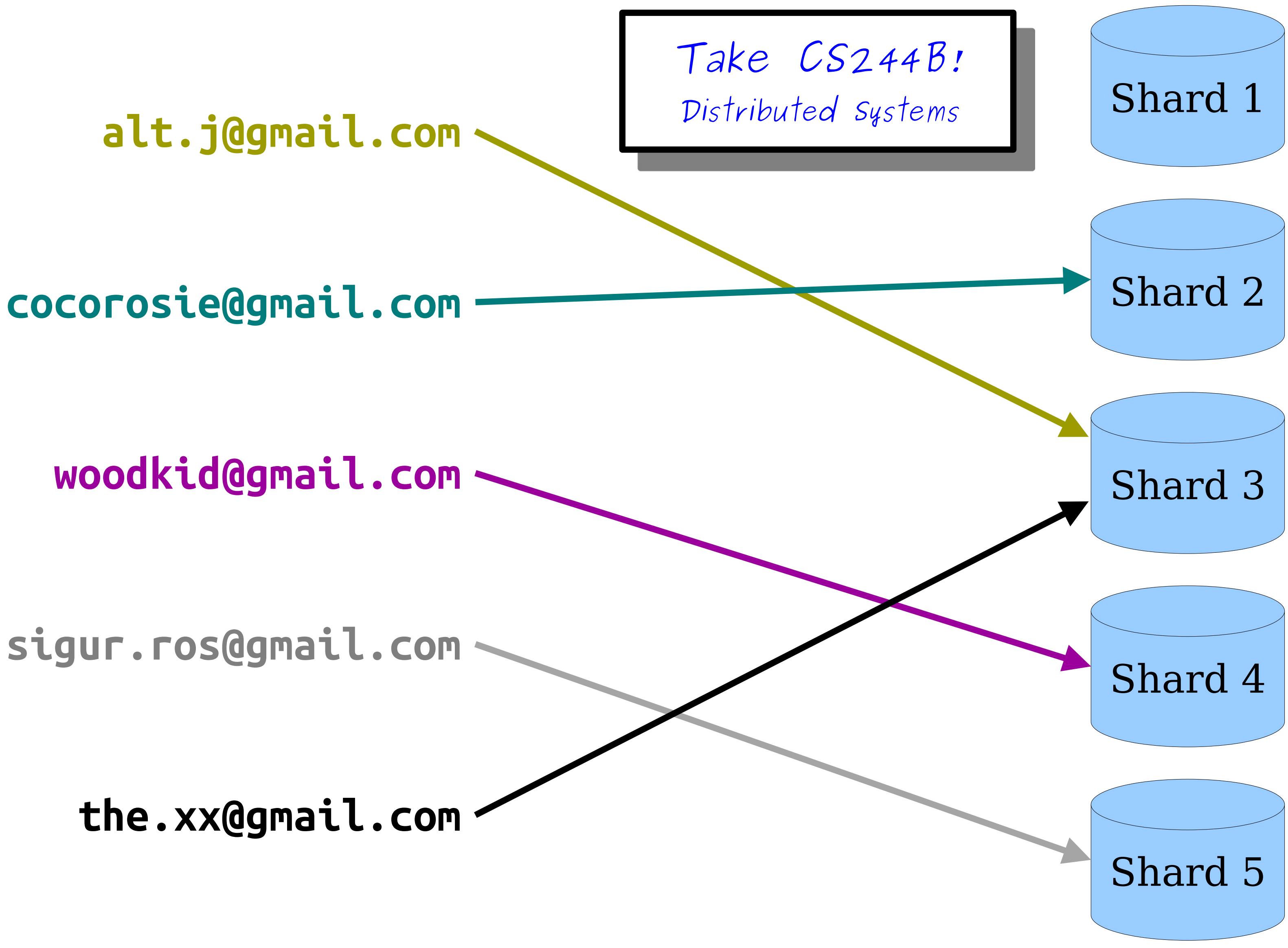
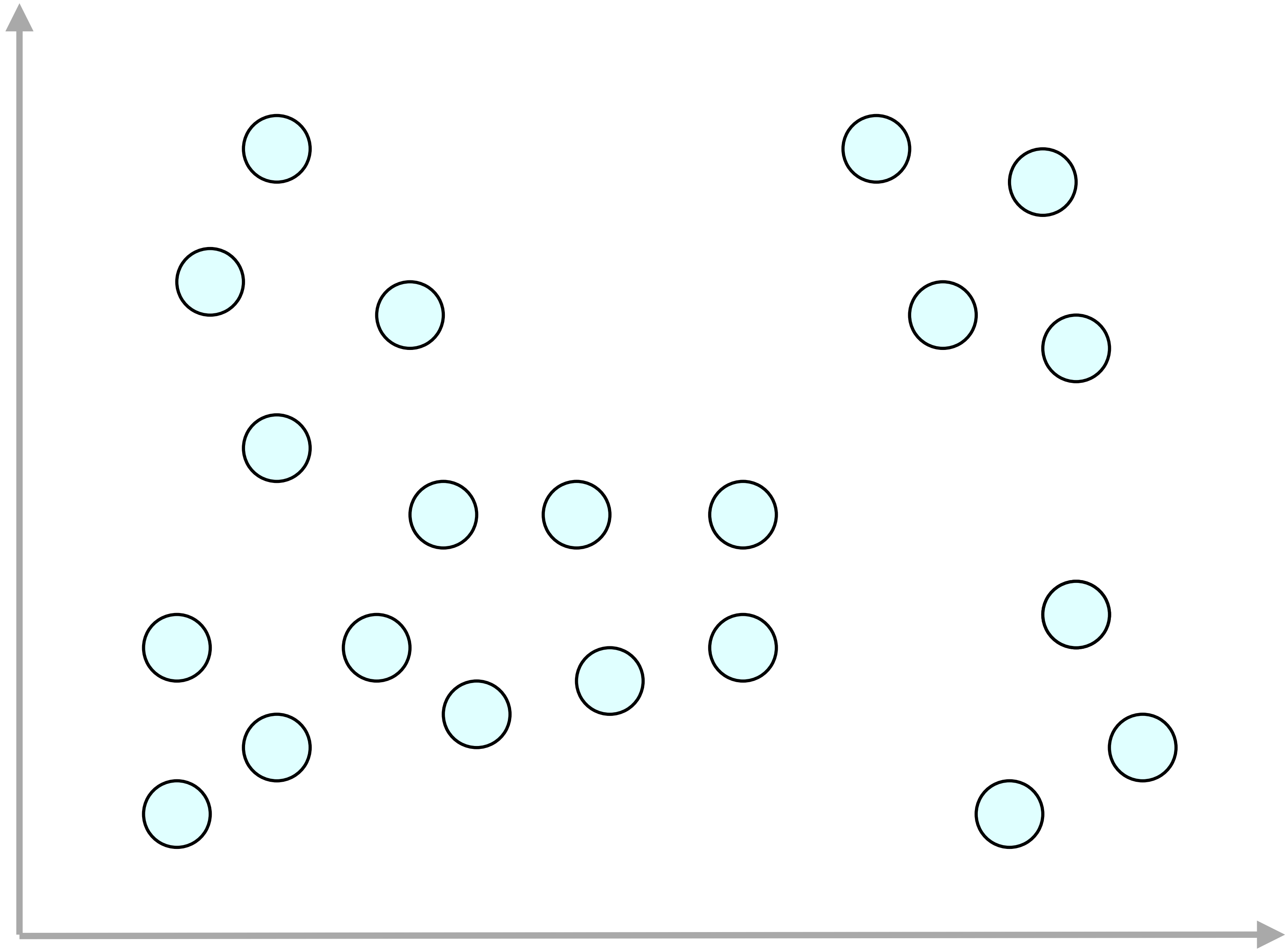


Example: Data Clustering



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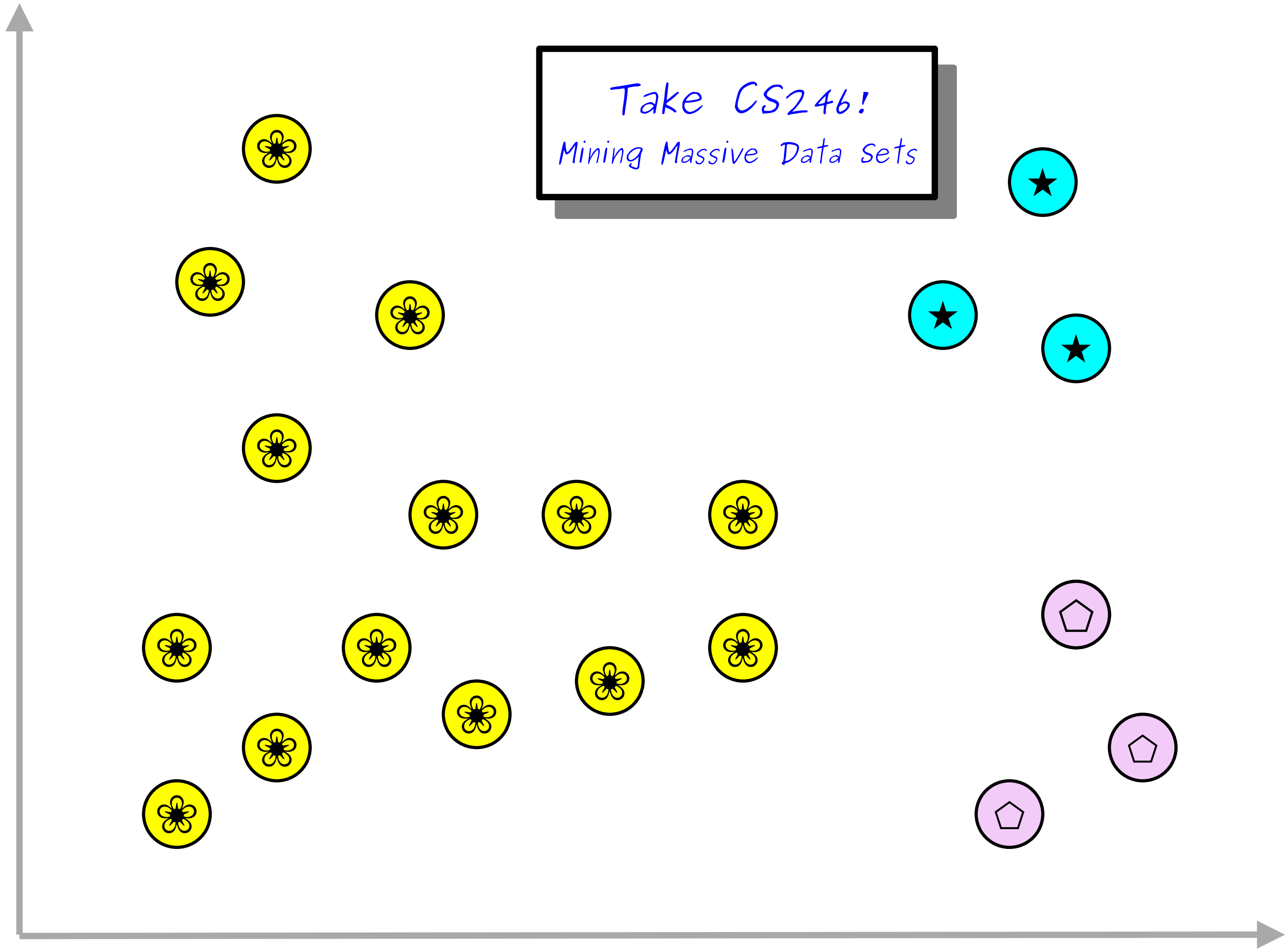
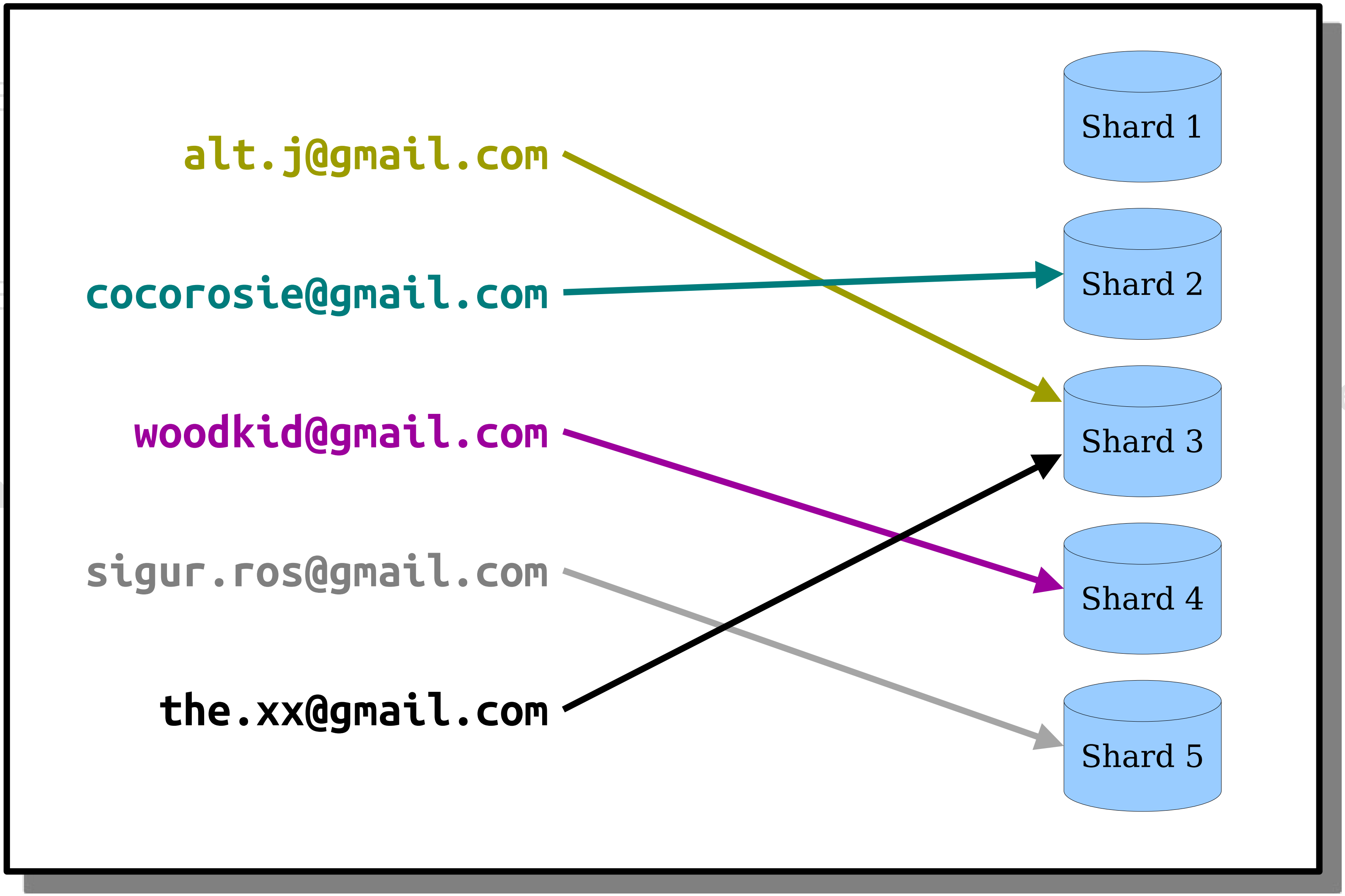


What's In Common?

- We have a **fixed, known set of possible inputs**.
  - In our examples: user names and 2D data points.
- We have a **fixed, known set of possible outputs**.
  - In our examples: database shards and cluster labels.
- **Each input** is assigned an output.
  - Some outputs might be assigned multiple inputs.
  - Some outputs might be assigned no inputs.

What's In Common?



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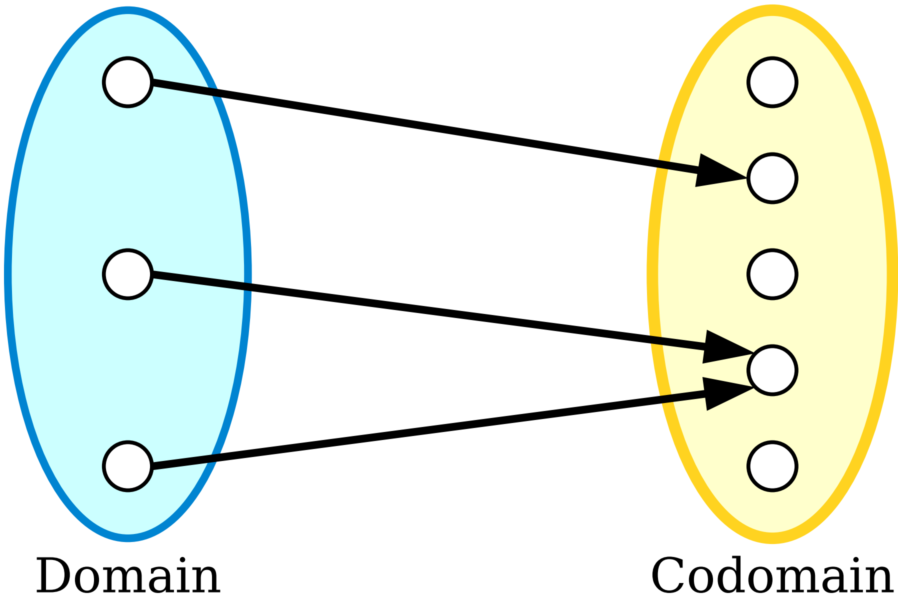
Domains and Codomains

- Every function *f* has two sets associated with it: its ***domain*** and its ***codomain***.
- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.

The function must be defined for every element of the domain.



The output of the function must always be in the codomain, but not all elements of the codomain must be produced as outputs.

Domains and Codomains

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- A function *f* can only be applied to elements of its domain. For any *x* in the domain, *f*(*x*) belongs to the codomain.

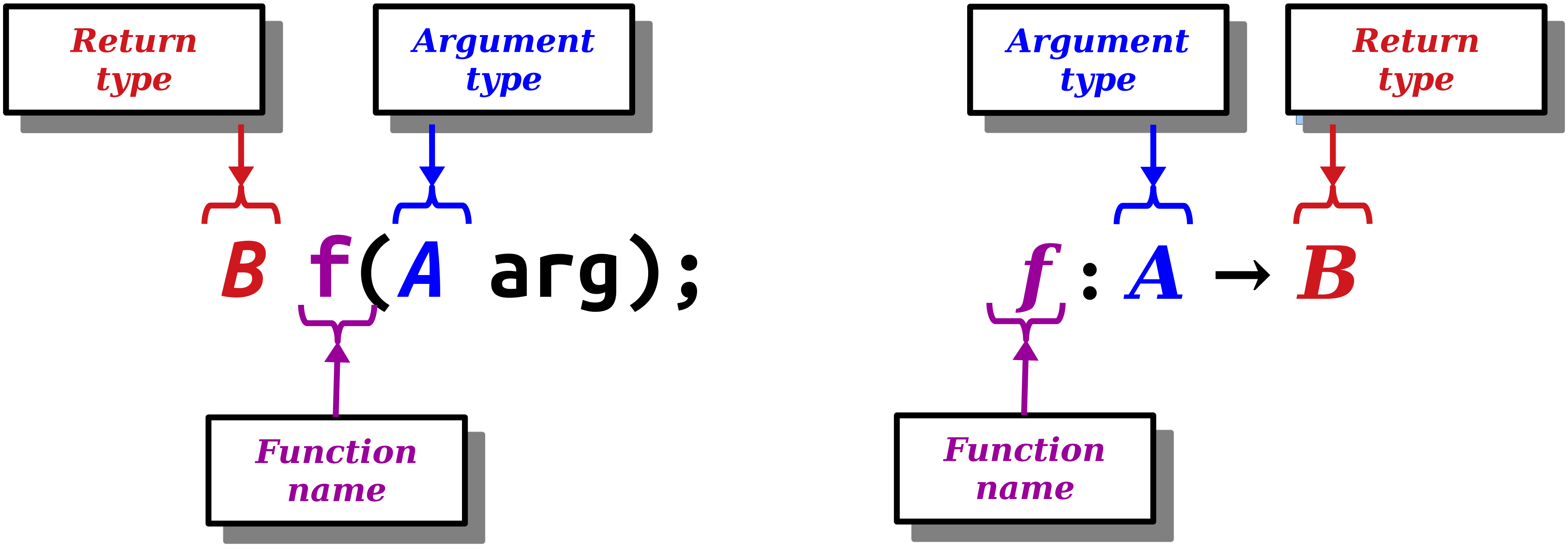
The ***domain*** of this function is ℝ. Any real number can be provided as input.

The ***codomain*** of this function is ℝ. Everything produced is a real number, but not all real numbers can be produced.

```
double absoluteValueOf(double x) {
    if (x >= 0) {
        return x;
    } else {
        return -x;
    }
}
```

Domains and Codomains

- If f is a function whose domain is A and whose codomain is B, we write $\boldsymbol{f : A \to B}$.
- Think of this like a "function prototype" in C++.



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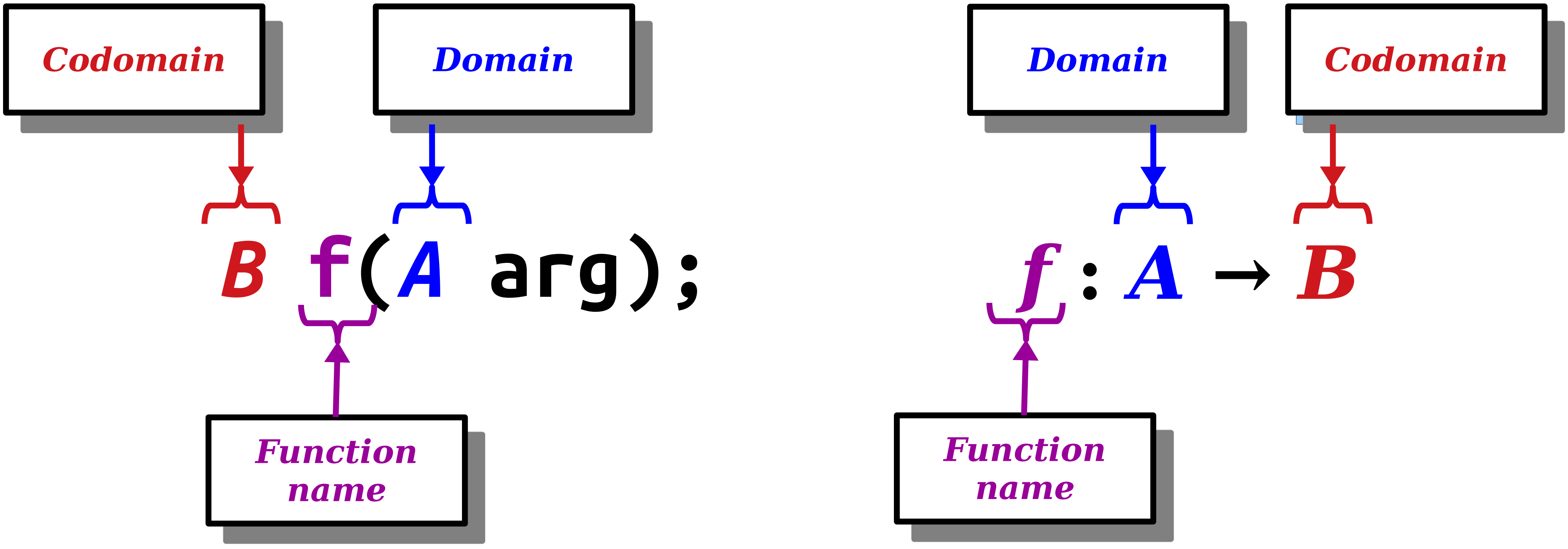

Some Observations

- Usually, when working with functions, you pick the domain and codomain before defining the rule for the function.
  - Think programming: you usually know what types of things you're working with before you know how they work.
- In mathematics, all functions take in exactly one argument: an element of the domain.
  - If you're clever, you can get two or more arguments to a function while still obeying this rule. Chat with me after class to learn more!
- In mathematics, functions are ***deterministic*** and can't behave randomly.
  - If you're clever, you can get functions that kinda sorta ish look random. Chat with me after class to learn more!

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The Official Rules for Functions

- Formally speaking, we say that $f : A \to B$ if the following two rules hold.
- First, f must be obey its domain/codomain rules:

$$\forall a \in A.\ \exists b \in B.\ f(a) = b$$

(*"Every input in A maps to some output in B."*)

- Second, f must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

(*"Equal inputs produce equal outputs."*)

- If you're ever curious about whether something is a function, look back at these rules and check! For example:
  - Can a function have an empty domain?
  - Can a function have an empty codomain?

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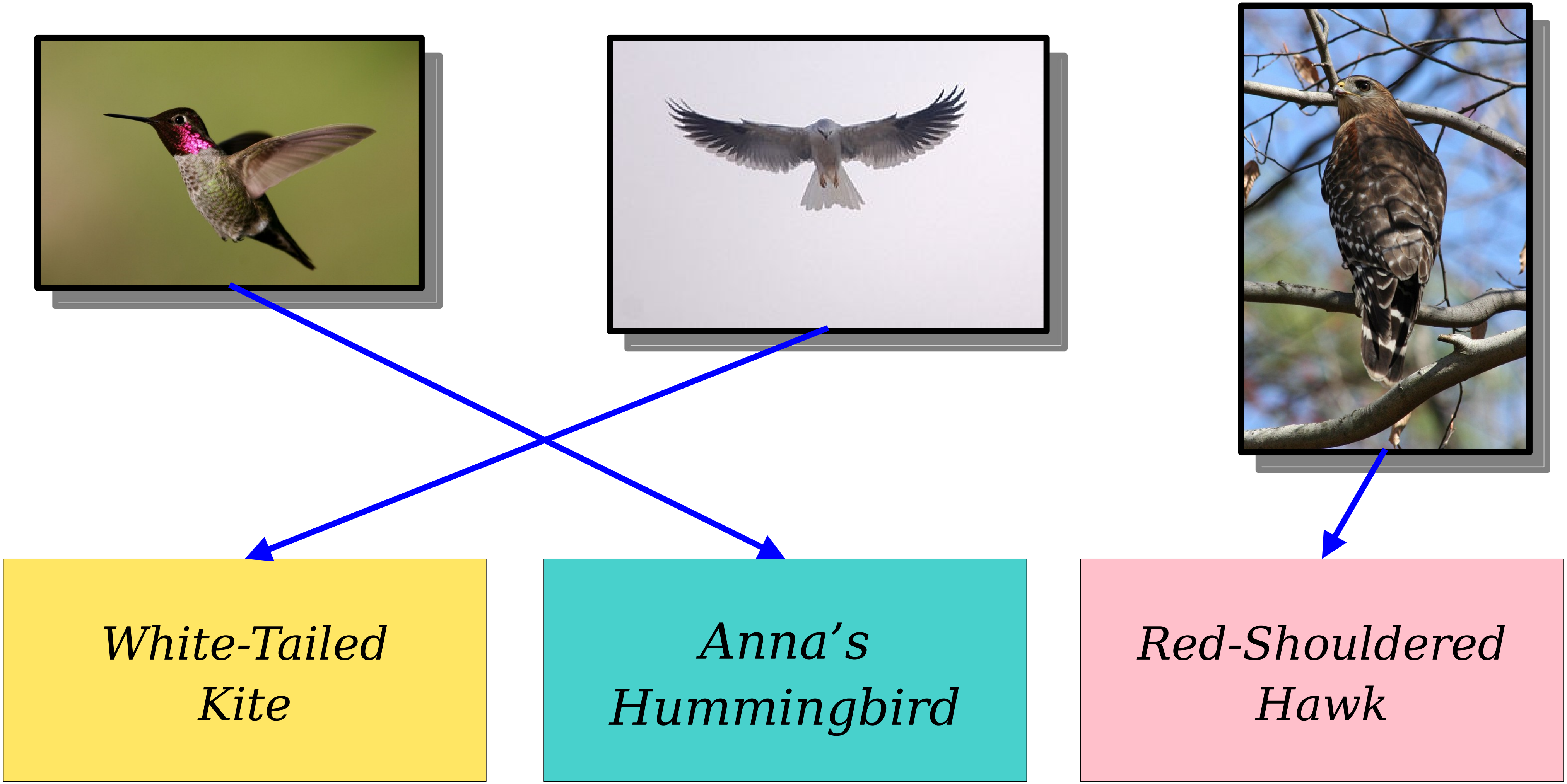
Defining Functions

- To define a function, you need to
  - specify the domain,
  - specify the codomain, and
  - give a ***rule*** used to evaluate the function.
- All three pieces are necessary.
  - We need to domain to know what the function can be applied to.
  - We need to codomain to know what the output space is.
  - We need the rule to be able to evaluate the function.
- There are many ways to do this. Let’s go over a few examples.

Defining Functions



Functions can be defined as a ***picture***. Draw the domain and codomain explicitly. Then, add arrows to show the outputs.

Defining Functions

$f : \mathbb{Z} \to \mathbb{Z}$, where

$$f(x) = x^2 + 3x - 15$$

Functions can be defined as a ***rule***.
Be sure to explicitly state what the domain and codomain are!

Defining Functions

$f : \mathbb{Z} \to \mathbb{N}$, where

$$f(n) = \begin{cases} n & \text{if } n \geq 0 \\ -n & \text{if } n \leq 0 \end{cases}$$

Some rules are given ***piecewise***. We select which rule to apply based on the conditions on the right. (Just make sure at least one condition applies and that all applicable conditions give the same result!)

Some Nuances

$f : \mathbb{R} \to \mathbb{R}$, where

$$f(x) = \frac{x+2}{x+1}$$

Answer at

cs103.stanford.edu/pollev

Is this a function from $\mathbb{R}$ to $\mathbb{R}$?

Some Nuances

$f : \mathbb{R} \to \mathbb{R}$, where

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Answer at

cs103.stanford.edu/pollev

This expression isn't defined when x = -1, so f isn't defined over its full domain. We therefore don't consider it to be a function.

Is this a function from $\mathbb{R}$ to $\mathbb{R}$?

Some Nuances

$f : \mathbb{N} \to \mathbb{R}$, where

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Is this a function from $\mathbb{N}$ to $\mathbb{R}$?

Some Nuances

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$$f(x) = \frac{x+2}{x+1}$$

Answer at

[*cs103.stanford.edu/pollev*](cs103.stanford.edu/pollev)

Yep, it's a function! Every natural number maps to some real number.

Is this a function from $\mathbb{N}$ to $\mathbb{R}$?

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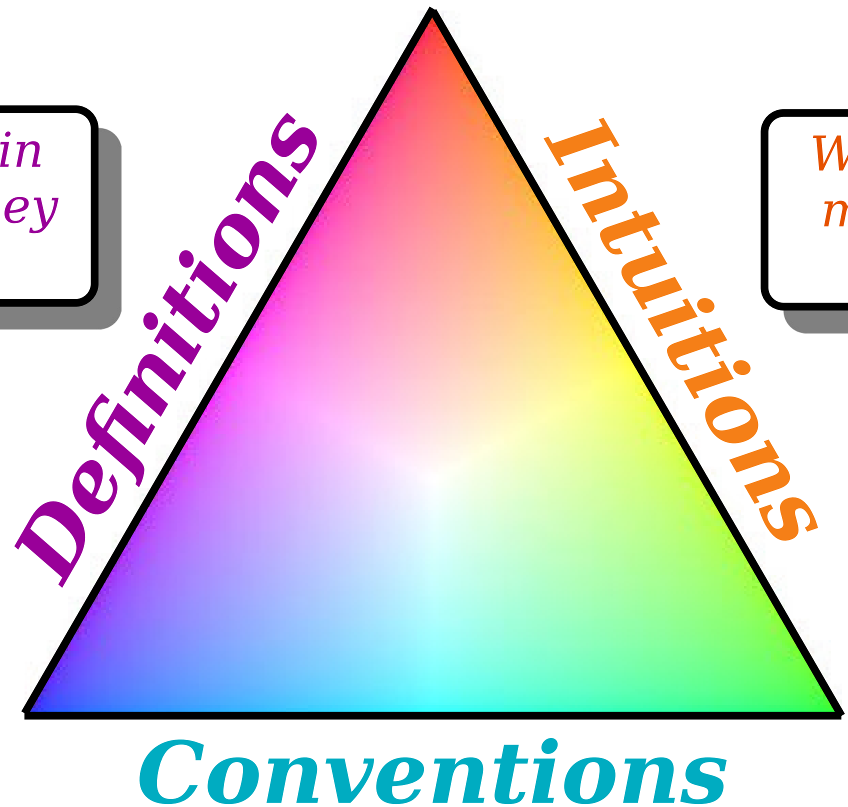
The Proof Writing Triad

What terms are used in this proof? What do they formally mean?

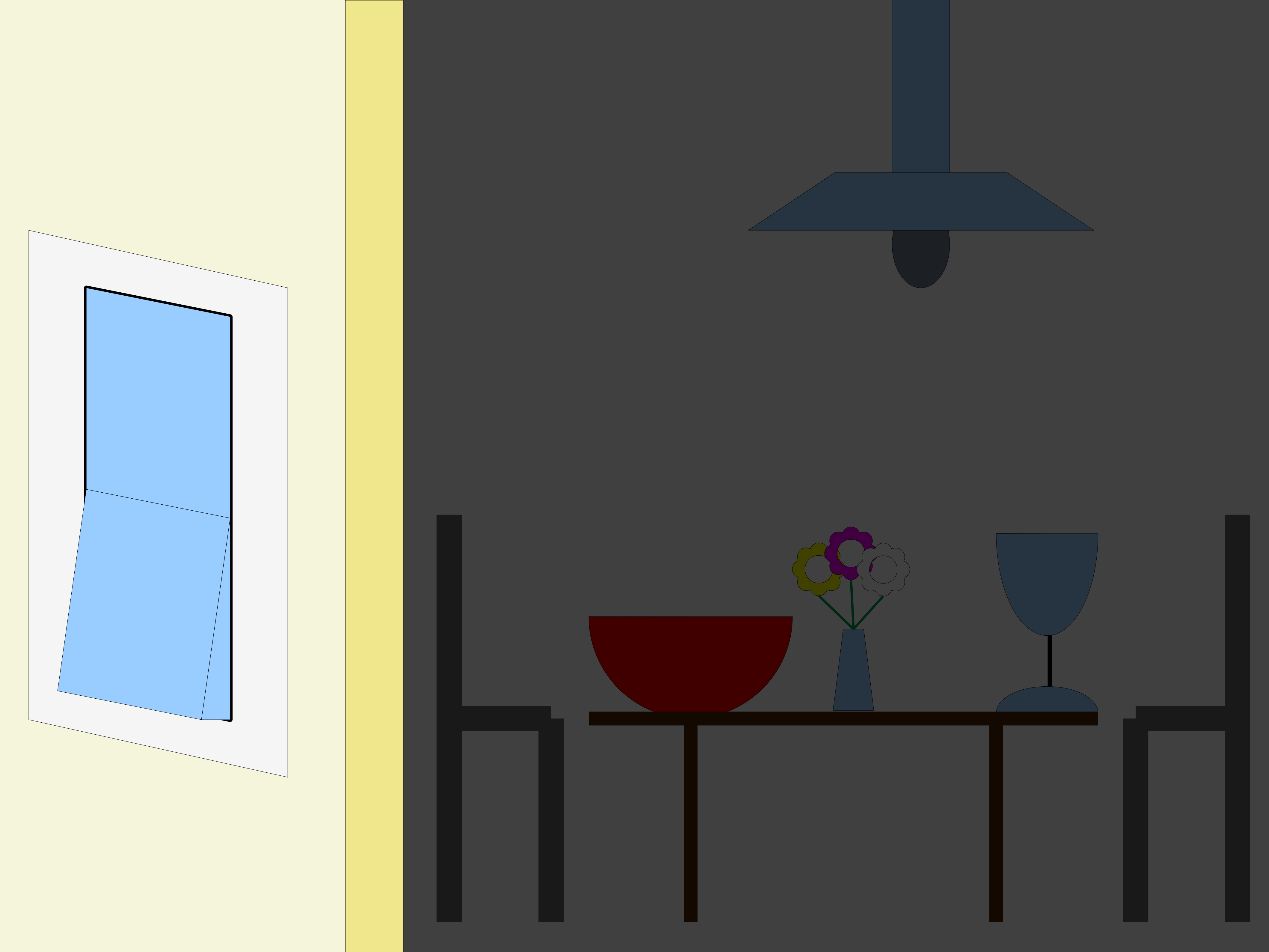


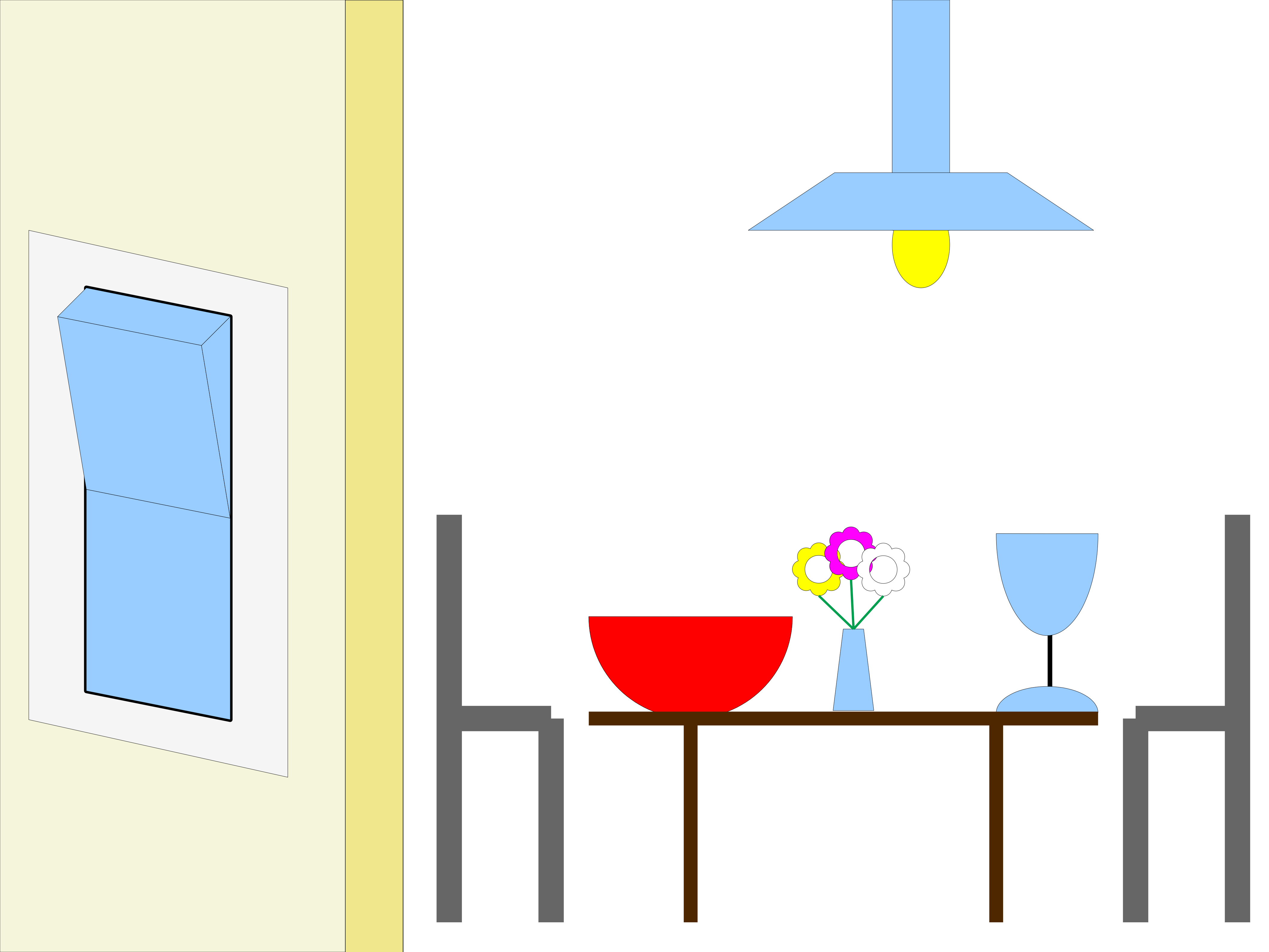
What does this theorem mean? Why, intuitively, should it be true?

What is the standard format for writing a proof? What are the techniques for doing so?

Undoing by Doing Again

- Some operations invert themselves. For example:
  - In first-order logic, $\neg\neg A$ is equivalent to A.
  - In algebra, $-(-x) = x$.
  - Flipping a switch twice is the same as not flipping it at all.





Undoing by Doing Again

- Some operations invert themselves. For example:
  - In first-order logic, $\neg\neg A$ is equivalent to A.
  - In algebra, $-(-x) = x$.
  - Flipping a switch twice is the same as not flipping it at all.
  - In set theory, $(A \Delta B) \Delta B = A$. *(Yes, really!)*
- Operations with these properties are surprisingly useful in CS theory and come up in a bunch of contexts.
  - Storing compressed approximations of sets (XOR filters).
  - Building encryption systems (symmetric block ciphers).
  - Transmitting a large file to multiple receivers (fountain codes).

Involutions

- A function $f : A \to A$ from a set back to itself is called an ***involution*** when the following first-order logic statement is true about f:

$$\forall x \in A.\ f(f(x)) = x.$$

("Applying f twice is equivalent to not applying f at all.")

- Involutions have lots of interesting properties. Let's explore them and see what we can find.

This is the formal definition. Use it in proofs.

This is just an intuition. Don't use it in proofs.

Involutions

- Which of the following are involutions?
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$.
  - $g : \mathbb{Z} \to \mathbb{Z}$ defined as $g(x) = -x$.
  - $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = {}^{1}/_{x}$.
  - $p : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$p(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

Answer at

cs103.stanford.edu/pollev

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  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = x$. *Yep!*
  - $g : \mathbb{Z} \to \mathbb{Z}$ defined as $g(x) = -x$. *Yep!*
  - $h : \mathbb{R} \to \mathbb{R}$ defined as $h(x) = {}^1/_x$.
  - $p : \mathbb{N} \to \mathbb{N}$ defined as follows:

$$p(n) = \begin{cases} n+1 & \text{if } n \text{ is even} \\ n-1 & \text{if } n \text{ is odd} \end{cases}$$

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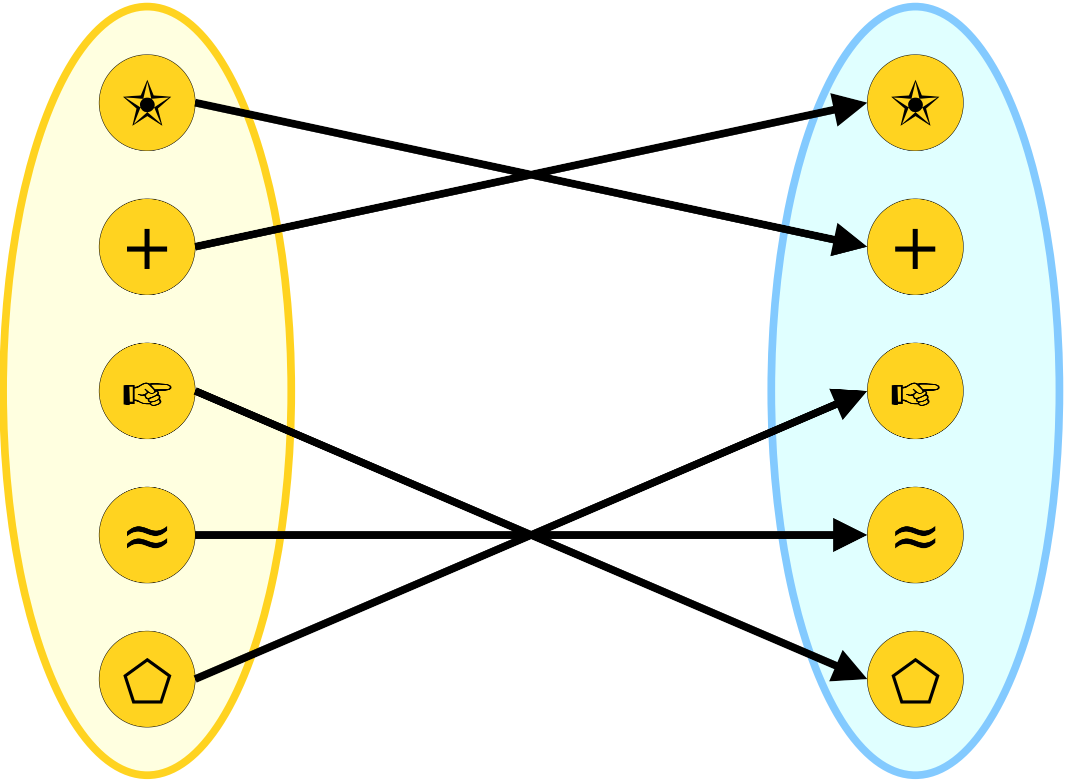

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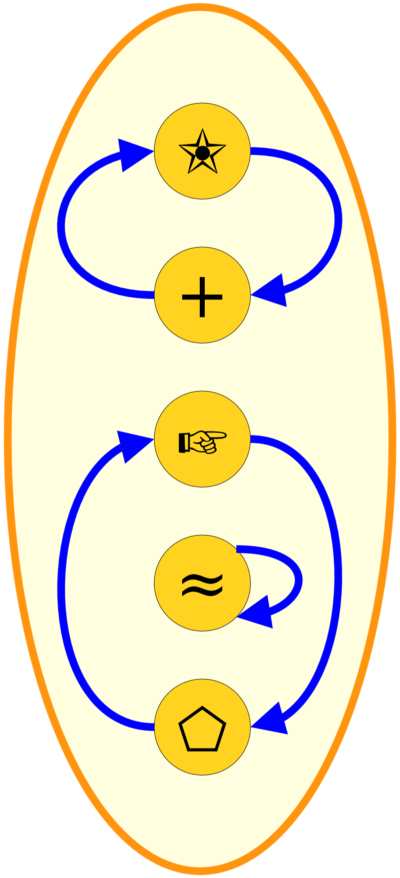

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Proofs on Involutions

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Case 2: n is odd.

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Functions

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A REALLY BIG DEAL!

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What does it mean for f to be an involution?

$$\forall n \in \mathbb{N}.\ f(f(n)) = n.$$

What is the negation of this statement?

$$\neg \forall n \in \mathbb{N}.\ f(f(n)) = n$$
$$\exists n \in \mathbb{N}.\ \neg(f(f(n)) = n)$$
$$\exists n \in \mathbb{N}.\ f(f(n)) \neq n$$

Therefore, we need to find some concrete choice of n such that $f(f(n)) \neq n$.

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Proof: We need to show that there is some $n \in \mathbb{N}$ where $f(f(n)) \neq n$.

Pick $n = 2$. Then

$$\begin{aligned} f(f(n)) &= f(f(2)) \\ &= f(4) \\ &= 16, \end{aligned}$$

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		To ***prove*** that this is true…
$\forall x.\ A$		Have the reader pick an arbitrary x. We then prove A is true for that choice of x.
$\exists x.\ A$		Find an x where A is true. Then prove A is true for that specific choice of x.
$\neg A$		Simplify the negation, then consult this table on the result.

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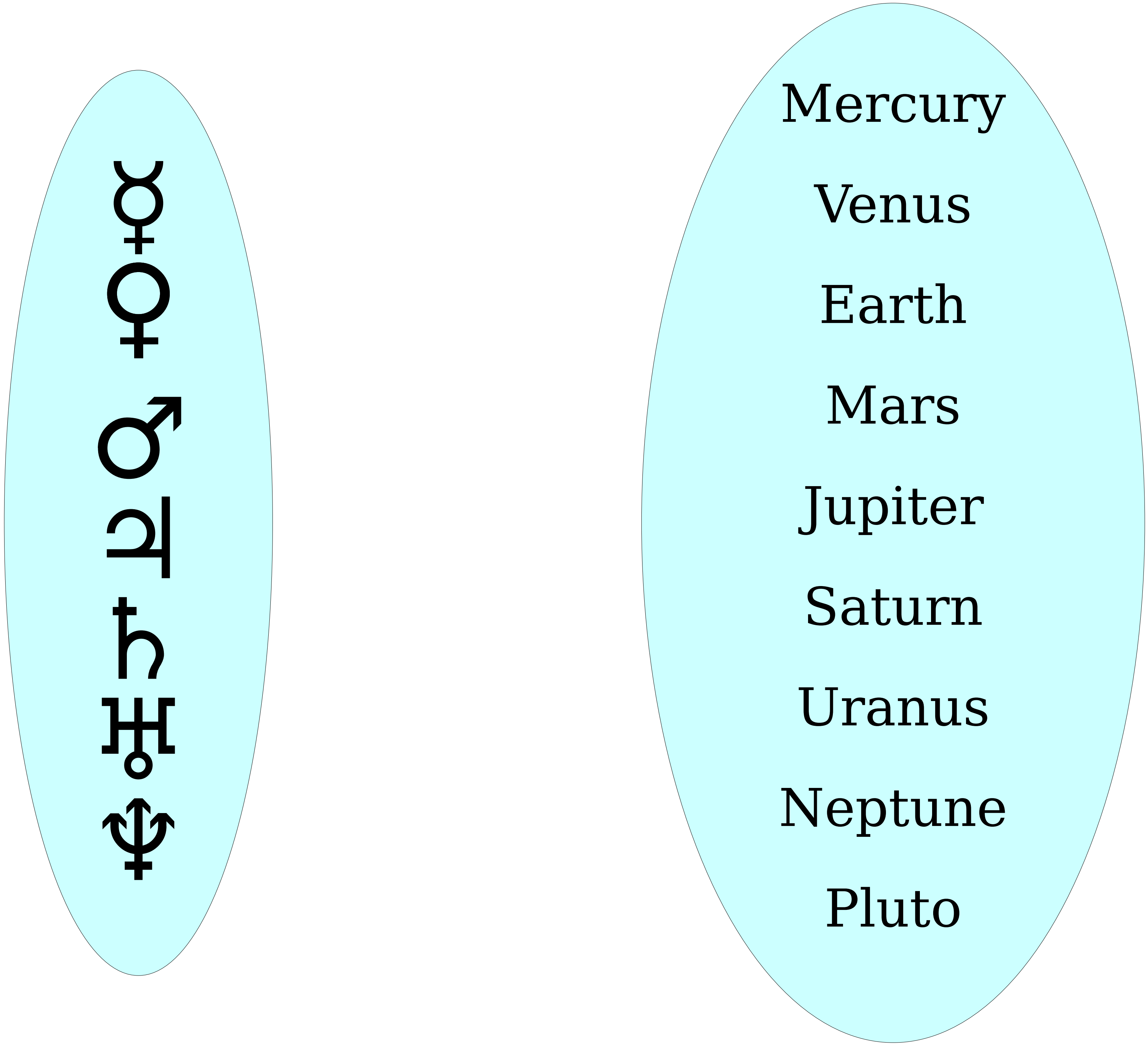
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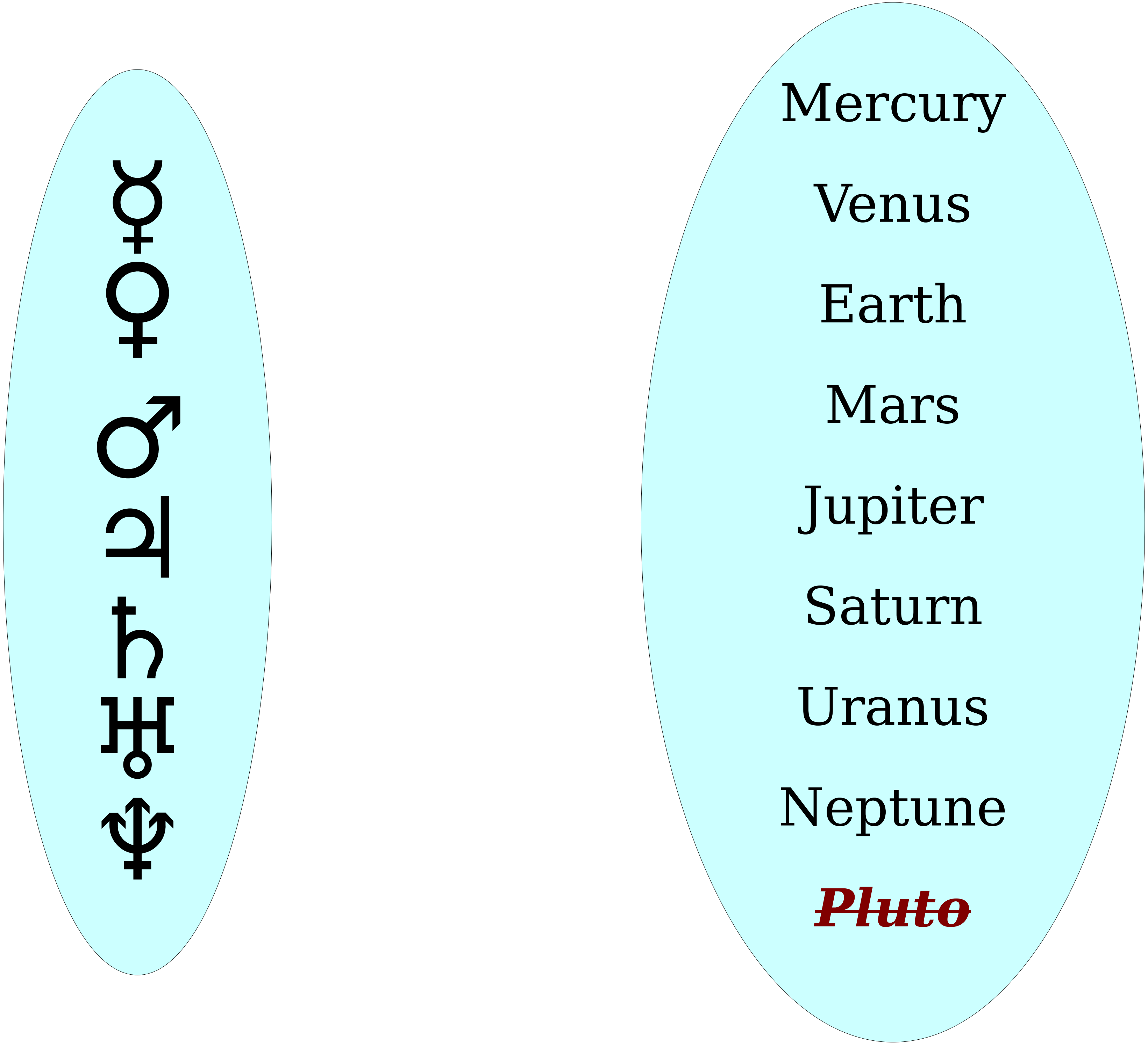
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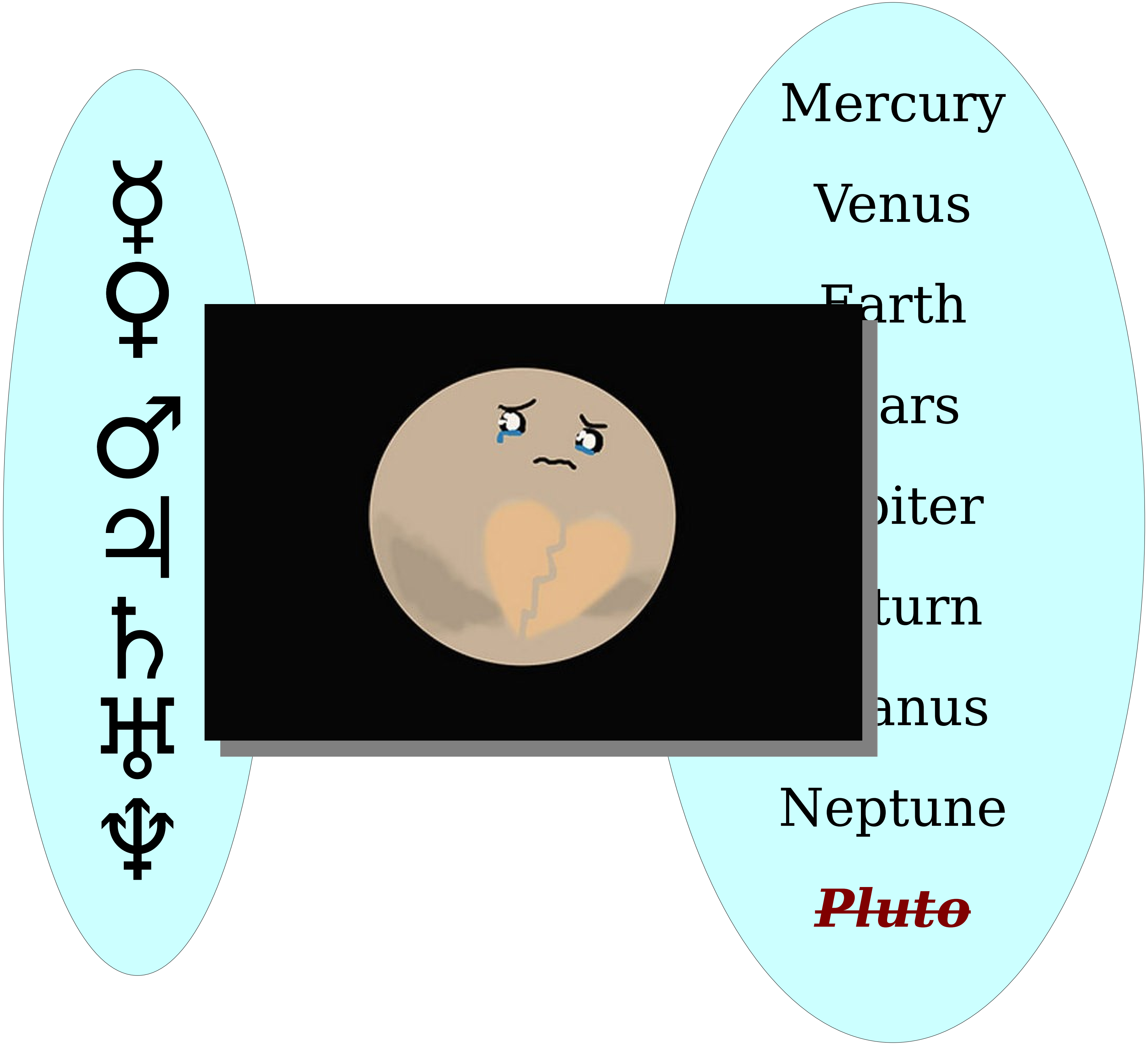
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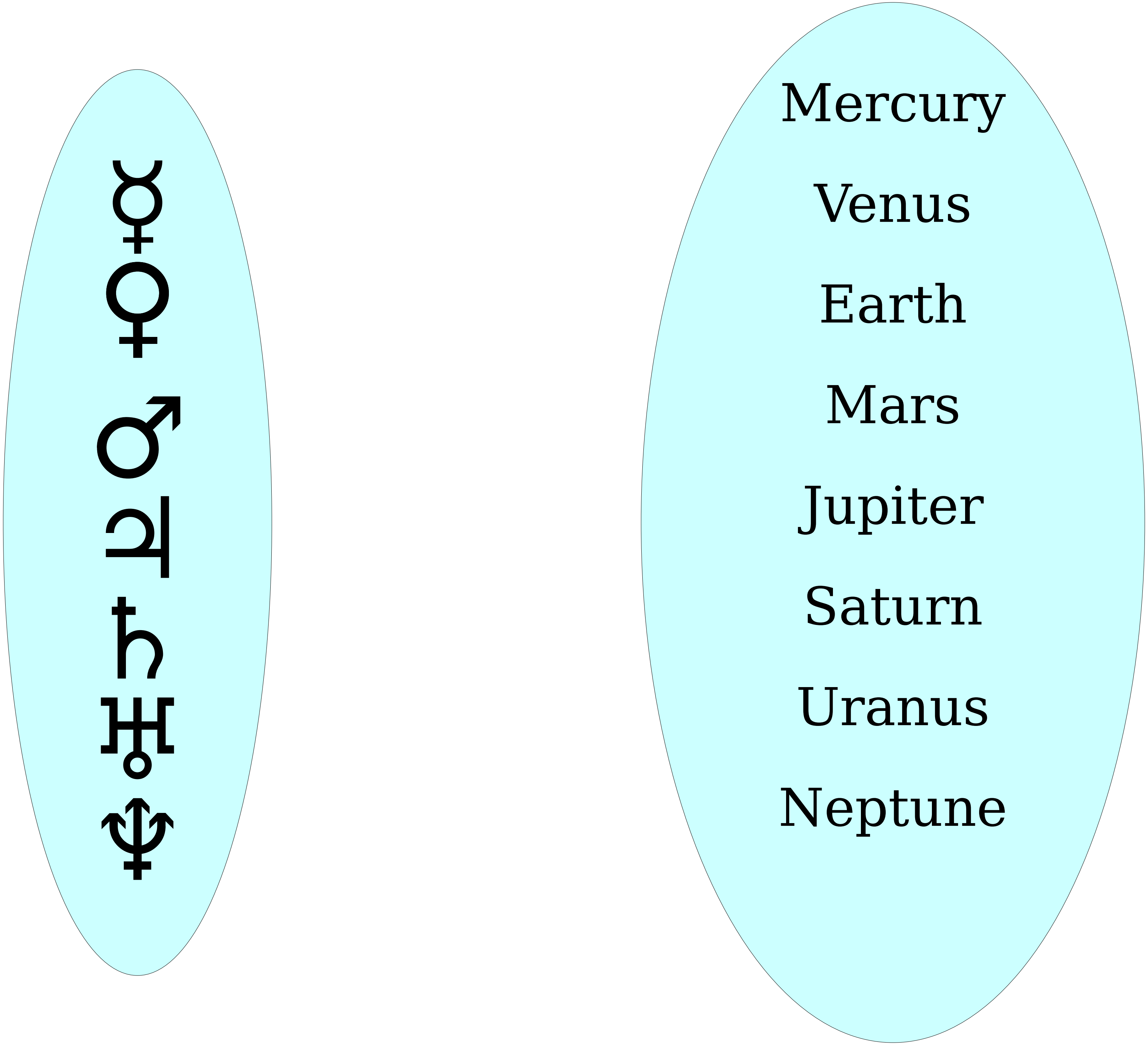
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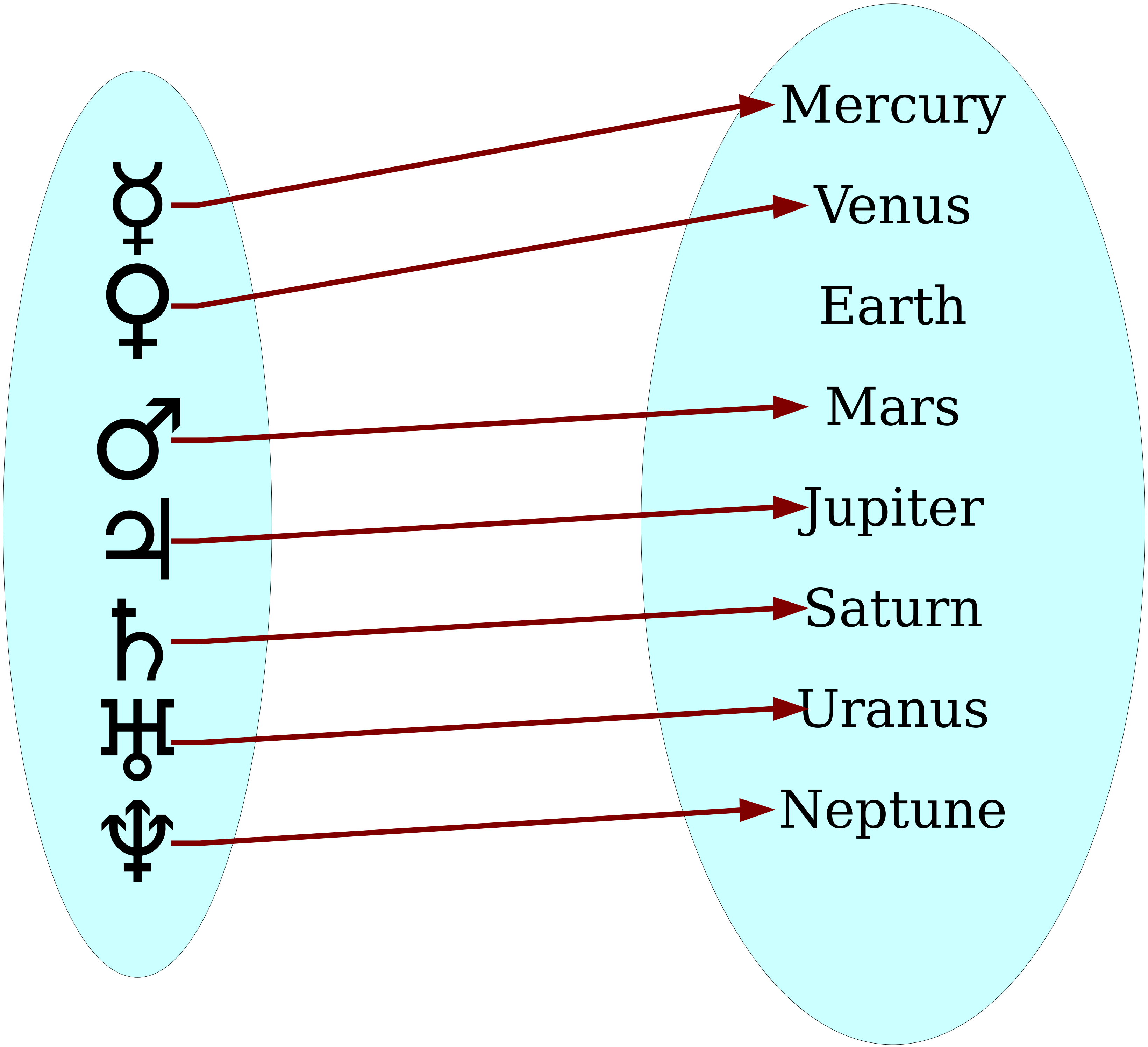

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Injective Functions

- A function $f : A \to B$ is called ***injective*** (or ***one-to-one***) when the following statement is true about f:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 \neq a_2 \to f(a_1) \neq f(a_2))$$

(*"If the inputs are different, the outputs are different."*)

- The following first-order definition is equivalent *(why?)* and is often useful in proofs.

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (f(a_1) = f(a_2) \to a_1 = a_2)$$

(*"If the outputs are the same, the inputs are the same."*)

- A function with this property is called an ***injection***.
- How does this compare to our second rule for functions?

Injec

Second, f must be deterministic:

$$\forall a_1 \in A.\ \forall a_2 \in A.\ (a_1 = a_2 \to f(a_1) = f(a_2))$$

(*"Equal inputs produce equal outputs."*)

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Injections

- Let be the set of all CS103 students. Which of the following are injective?
  - f : → ℕ where $f(x)$ is x's Stanford ID number.
  - g : → , where is the set of all continents and $g(x)$ is x's continent of birth.
  - h : → , where is the set of all given (first) names, where $h(x)$ is x's given (first) name.

Answer at

cs103.stanford.edu/pollev

$f : A \to B$ is ***injective*** when either equivalent statement is true:

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (x_1 \neq x_2 \to f(x_1) \neq f(x_2))$$

$$\forall x_1 \in A.\ \forall x_2 \in A.\ (f(x_1) = f(x_2) \to x_1 = x_2)$$

Injections

- Let be the set of all CS103 students. Which of the following are injective?

✓ - f : $\to \mathbb{N}$ where $f(x)$ is x's Stanford ID number.

- g : $\to$, where is the set of all continents and $g(x)$ is x's continent of birth.

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Proofs on Injections

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Good exercise: Repeat this proof using the other definition of injectivity!

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What does it mean for f to be injective?

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What is the negation of this statement?

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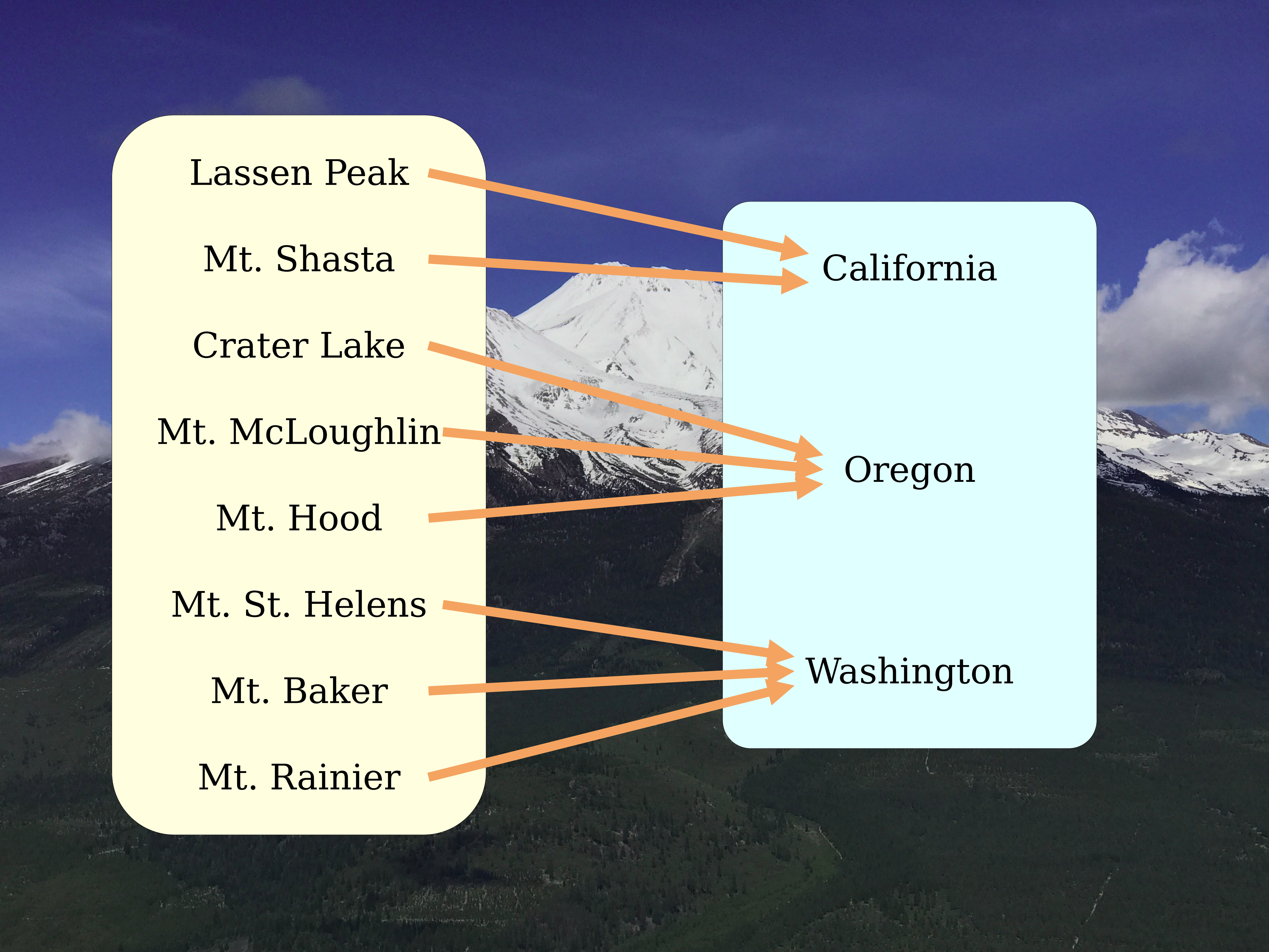
Functions

Part 1

1. Announcements
2. Functions: Intuitions and Examples
3. Domains and Codomains
4. Official Rules for Functions
5. Ways to Define Functions
6. Involutions (with Proofs!)
7. The Assume-Prove Table
8. **Injections, Surjections, and Bijections**
9. What's Next?

Still Here


Lassen Peak
Mt. Shasta
Crater Lake
Mt. McLoughlin
Mt. Hood
Mt. St. Helens
Mt. Baker
Mt. Rainier
California
Oregon
Washington

Surjective Functions

- A function $f : A \to B$ is called ***surjective*** (or ***onto***) when this first-order logic statement is true about f:

$$\forall b \in B.\ \exists a \in A.\ f(a) = b$$

(*"For every possible output,
there's an input that produces it."*)

- A function with this property is called a ***surjection***.
- How does this compare to our first rule of functions?

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- How does this compare to our first rule of functions?

First, f must be obey its domain/codomain rules:

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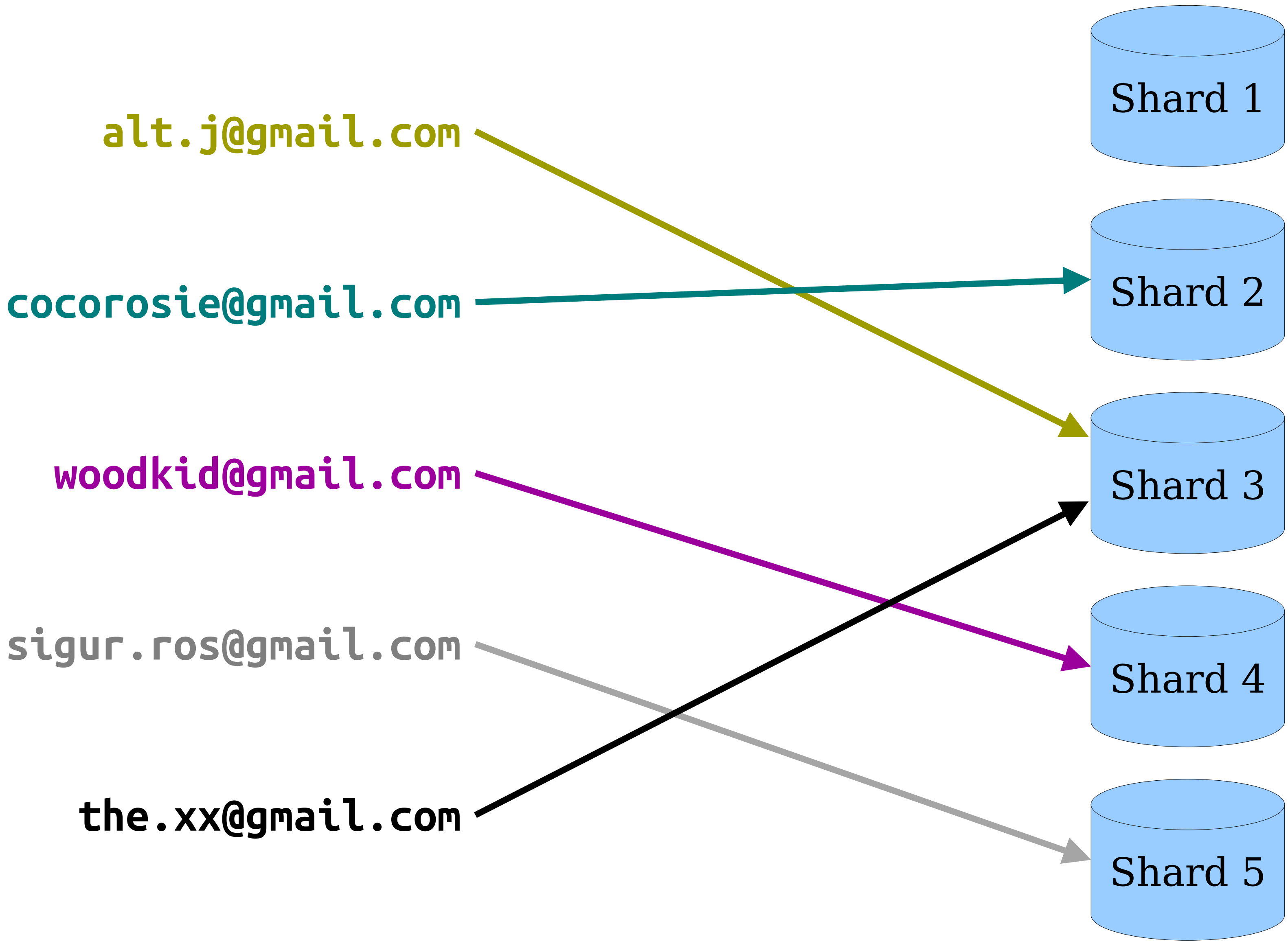
(*"Every input in A maps to some output in B."*)

Example: Database Sharding



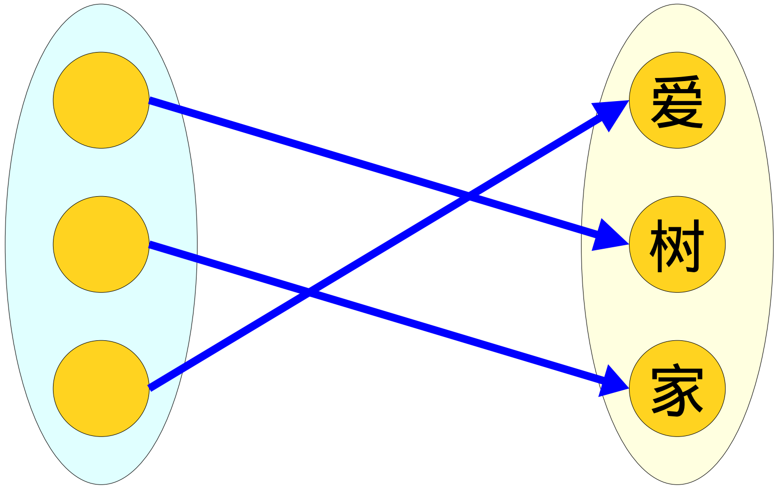

See appendix for sample proofs involving surjections.

Injections and Surjections

- An injective function associates ***at most*** one element of the domain with each element of the codomain.
- A surjective function associates ***at least*** one element of the domain with each element of the codomain.
- What about functions that associate ***exactly one*** element of the domain with each element of the codomain?

Bijections

- A ***bijection*** is a function that is both injective and surjective.
- Intuitively, if $f : A \to B$ is a bijection, then f represents a way of pairing off elements of A and elements of B.



Bijections

- Which of the following are bijections?
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = x$.
  - $f : \mathbb{Z} \to \mathbb{R}$ defined as $f(x) = x$.
  - $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = 2x + 1$.
  - $f : \mathbb{Z} \to \mathbb{Z}$ defined as $f(x) = 2x + 1$.

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Next Time

- ***First-Order Assumptions***
  - The difference between assuming something is true and proving something is true.
- ***Connecting Function Types***
  - Involutions, injections, and surjections are related to one another. How?
- ***Function Composition***
  - Sequencing functions together.

Appendix:

More Proofs on Functions

Proof 1: Proving a function is surjective.

Surjective Functions

Theorem: Let $f : \mathbb{R} \to \mathbb{R}$ be defined as $f(x) = 2x$. Then $f(x)$ is surjective.

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Proof:

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Proof:

What does it mean for f to be surjective?

$$\forall y \in \mathbb{R}.\ \exists x \in \mathbb{R}.\ f(x) = y$$

Therefore, we'll choose an arbitrary $y \in \mathbb{R}$, then prove that there is some $x \in \mathbb{R}$ where $f(x) = y$.

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Let $x = y / 2$.

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Let $x = y / 2$. Then we see that

$$f(x) = f(y / 2)$$

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This proof contains no first-order logic syntax (quantifiers, connectives, etc.). It's written in plain English, just as usual.

Proof 2: Proving a function is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

What does it mean for g to be surjective?

$$\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$

What is the negation of the above statement?

$$\neg\forall n \in \mathbb{N}.\ \exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \neg\exists m \in \mathbb{N}.\ g(m) = n$$
$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n$$

Therefore, we need to find a natural number n where, regardless of which m we pick, we have $g(m) \neq n$.

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Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

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Surjective Functions

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Proof: Let $n = 137$.

Our overall goal is to prove

$$\exists n \in \mathbb{N}.\ \forall m \in \mathbb{N}.\ g(m) \neq n.$$

We just made our choice of n. Therefore, we need to prove

$$\forall m \in \mathbb{N}.\ g(m) \neq n.$$

We'll therefore pick an arbitrary $m \in \mathbb{N}$, then prove that $g(m) \neq n$.

Surjective Functions

Theorem: Let $g : \mathbb{N} \to \mathbb{N}$ be defined as $g(n) = 2n$. Then $g(x)$ is not surjective.

Proof: Let $n = 137$. Now, pick an arbitrary $m \in \mathbb{N}$.

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Notice that $g(m) = 2m$ is even, while 137 is odd.

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